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ON CIRCULAR TRANSFORMATIONS.

By Dr. Arnold Emch, Biel, Switzerland.

One of the peculiar characteristics of Steinerian methods is the simplicity of the fundamental principles on which these methods are based. Starting from a simple proposition in elementary geometry Steiner, in the course of his investigations, often arrived at a general point of view from which it was easy to state a great number of theorems which, otherwise, would have been puzzles to mathematicians. To account for this fact, it must be remembered that Jacob Steiner in his early age was a pupil of Pestalozzi, so that, perhaps, the secret of Steiner's success lies in the influence of his great master.

In this paper I shall have reference to some of those theorems which Steiner derived from the so-called " power of a point " with regard to a circle (square of the tangent distance from the point to the circle) and which may be found in the first volume of his collected works under the title: *Einige geometrische Betrachtungen.** It will be seen that they naturally follow from the consideration of *Circular Transformations*, using the designation of Darboux† for all those conform transformations which transform circles into circles.

1. First I will confine myself to the plane of complex quantities. Two points z and z' correspond to each other in a circular transformation if they are connected by the fundamental equation

$$z' = \frac{az + b}{cz + d}, \tag{1}$$

where the constants a, b, c, d are real or imaginary. A circular transformation (1) may be found which transforms three points of the plane z into any other three points of the same plane (the planes z and z' are assumed coincident); consequently the circle through the first three points is transformed into the circle through the three transformed points. If in fig. 1 through the points z_1 and z_2 I choose a circle c_1, including a certain angle with the circle c through z_1, z_2, z_3, the corresponding circle c_1' through the points z_1' and z_2' is determined and includes the same angle with the circle c' through z_1', z_2', z_3'.

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* *Jacob Steiner's Gesammelte Werke*, Vol. I, pp. 17–76.

† Darboux: *La Théorie Générale des Surfaces*, Part I, page 162.

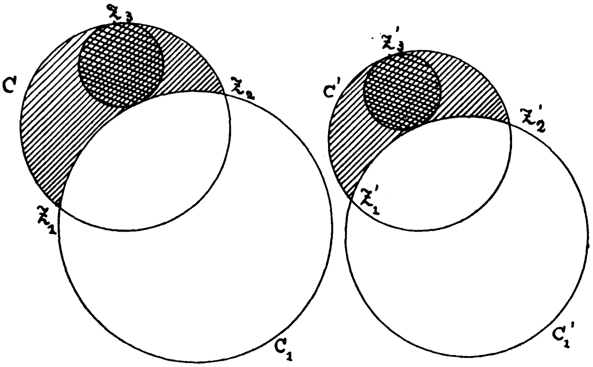

Fig. 1.

On the circle c any point z_3, and on the circle c' any point z_3' may be chosen, so that the new circular transformation is perfectly determined and still transforms the circles c and c_1 into c' and c_1', respectively. Keeping the point z_3 fixed, any point z_3' on the circumference c' may be chosen as corresponding to z_3. There are, therefore, ∞^1 circular transformations which transform the circles c and c_1 into c' and c_1', respectively. A circle tangent at z_3 to c and tangent to c_1, for a fixed transformation, is transformed into a circle tangent to c' at z_3' and tangent to c_1'. In general, a circle passing through z_3 and including certain angles with the circles c and c_1 is transformed into a circle passing through z_3' and including the same angles with the circles c' and c_1'. Now, as the choice of the points z_1, z_2, z_3, and the angles at these points is arbitrary, the theorem holds :

Any two circular triangles with corresponding equal angles may be considered as corresponding in a circular transformation.

There is only one transformation of this kind.

If all the angles are zero the theorem is :

There is only one circular transformation which transforms a circular triangle with all its angles equal to zero into any other fixed triangle with the angles zero.

Taking in fig. 2 a circle c_2 through z_1 normal to the circles c' and c_1 and intersecting these circles respectively in z_2 and z_3, an infinite number (∞^1) of circles c_2' through z_1' normal to the circles c' and c_1' may be passed successively corresponding to the circle c_2; hence, there are also in this special case ∞^1 circular transformations transforming the circles c and c_1 into c' and c_1'. One of Steiner's configurations is now obtained by drawing in fig. 3 a system of tangent circles in such a manner, that each of these circles touches the circles c and c_1 and its two neighboring circles.

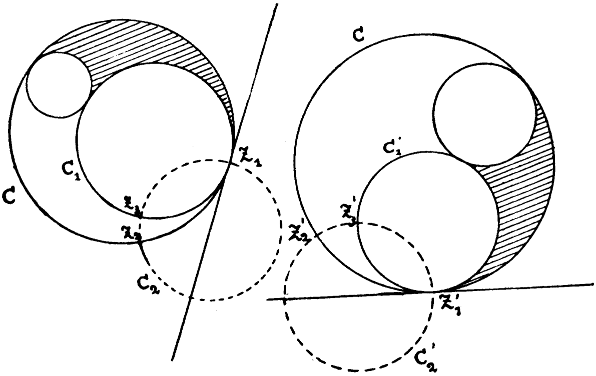

FIG. 2.

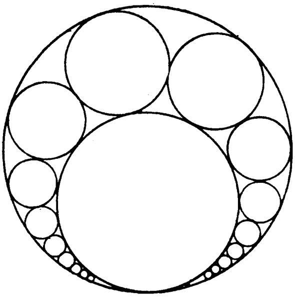

FIG. 3.

According to the foregoing theorem it is evident *that there is only one circular transformation which transforms one of these Steinerian configurations into any other fixed configurations of the same kind.* Among these figures the case must also be included where the circles c and c_1 become straight lines and parallel and all tangent-circles therefore equal. The following example will illustrate the relation between this and the general case.

2. In the plane z assume a system of equal circles so that each circle is tangent to six neighboring circles (fig. 3′). The centers of these circles form a net of rhombs whose angles are 60° and 120°. Assuming the origin in one of the centers and the axes x and y as indicated in figure 3′, the rectilinear coordinates α, β of any of these centers may be expressed in the form

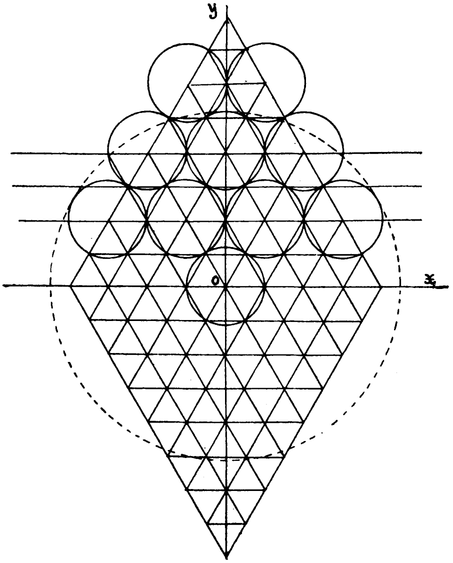

FIG. 3′.

$$\begin{aligned} \alpha &= \frac{1}{2} md + nd = \frac{d}{2}(m + 2n) \\ \beta &= \frac{d}{2} \sqrt{3} \, . \, m , \end{aligned} \tag{2}$$

where d designates the constant diameter of the circles and m and n any integral real numbers. Putting

$$A = \alpha + i\beta = \frac{d}{2}(m + 2n) + i \frac{d}{2} \sqrt{3} m ,$$

$$A_0 = \alpha - i\beta = \frac{d}{2}(m + 2n) - i \frac{d}{2} \sqrt{3} m ,$$

$$B = \alpha^2 + \beta^2 - \frac{d^2}{4} = d^2 \left[m^2 + mn + n^2 - \frac{1}{4} \right]$$

$$z_1 = x + iy,$$

$$z_{1,0} = x - iy,$$

the equation of any of the circles of the system may be written

$$z_1 z_{1,0} + A z_1 + A_0 z_{1,0} + B = 0\,. \tag{3}$$

Applying the inversion

$$\begin{aligned} z_1 &= \frac{1}{z}\,, \\ z_{1,0} &= \frac{1}{z_0}\,, \end{aligned} \tag{4}$$

equation (3) assumes the form

$$z z_0 + \frac{A_0}{B} z + \frac{A}{B} z_0 + \frac{1}{B} = 0\,.$$

This equation may represent any circle of the inversed system of (3). Designating generally the rectilinear coordinates of such a circle by a and b and the radius by r, we find

$$\frac{A_0}{B} = a + ib \quad , \quad \frac{A}{B} = a - ib\,,$$

hence

$$a = \frac{A + A_0}{2B} = \frac{\alpha}{B} \quad , \quad b = \frac{A_0 - A}{2iB} = -\frac{\beta}{B}\,,$$

or after some reductions

$$\begin{aligned} a &= \frac{m + 2n}{2d\,(m^2 + mn + n^2 - \frac{1}{4})}\,, \\ b &= \frac{-m\sqrt{3}}{2d\,(m^2 + mn + n^2 - \frac{1}{4})}\,, \\ r &= \frac{1}{2d\,(m^2 + mn + n^2 - \frac{1}{4})}\,. \end{aligned} \tag{6}$$

Giving in these expressions m and n all integral values between $-\infty$ and $+\infty$, the inversed system of (3) arises. It is interesting to remark, that the ratios $\frac{a}{r}$ and $\frac{b}{r}$ depend only on m and n. Considering especially the ratio

$$\frac{a}{r} = m + 2n\,, \tag{7}$$

we can derive immediately the propositions which Pappus gave in his *Collectiones Mathematicæ, libr. IV* ("*circumfertur in quibusdam libris antiqua propositio huiusmodo.*")*

Assuming in (7) m constant it may first be assumed that m is an even number, say $m = 2l$, then

$$\frac{a}{r} = 2(l + n).$$

For $n = -l$ there is $\frac{a}{r} = 0$. Putting successively for n the values, $-l$, $1 - l$, $2 - l$, ..., the corresponding values of the ratio $\frac{a}{r}$ are obtained from the following table:

n	$-l$	$1-l$	$2-l$	$3-l$	...
$\frac{a}{r}$	0	2	4	6	...

The circles corresponding to this series form a Steinerian configuration and in fig. 4, which illustrates half of the inversed system of (3) (size ½ of what it ought to be), are represented by the white circles between the two shaded series.

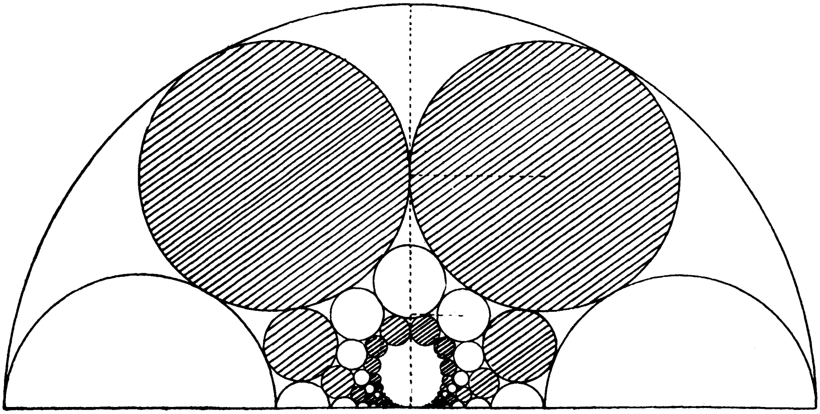

FIG. 4.

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* See Steiner's Werke, Vol. I, p. 47.

If m is an odd number, $m = 2l + 1$,

$$\frac{a}{r} = 2l + 1 + 2n .$$

Here the smallest ratio is 1 which results, when $n = -l$, and the table for $n = -l, 1 - l, 2 - l, \ldots$, is

n	$-l$	$1-l$	$2-l$	$3-l$	$\ldots$
$\frac{a}{r}$	1	3	5	7	$\ldots$

The corresponding circles form again a Steinerian configuration, and in fig. 4 either series of shaded circles may represent them.

The proposition of Pappus, to which he refers as an " old theorem," consists in the fact that in both cases the ratios $\frac{a}{r}$ of consecutive circles form arithmetical series. It will be remarked that the theorem holds only for those positions of Steinerian configurations with regard to the axes x and y as indicated in fig. 3′. Generalizations and further details may be found in the article of Steiner mentioned above.

3. Three circles of which each two intersect, form 64 distinct circular triangles whose corners are the points of intersection and the sides arcs of these circles. We have seen that any two circular triangles whose corresponding angles are equal are corresponding in a circular transformation. This fact has been proved for real triangles, and it may be inferred that it also holds for imaginary and partly real and partly imaginary triangles.

For our purpose it is not necessary to give a direct proof for this case, for it will be shown that the condition for the correspondence of two circular triangles in a circular transformation can be expressed by means of Steinerian configurations in the case that the two fundamentals c and c_1 do not intersect* in real points.

First I will establish the preliminary theorems:

There are always circular transformations which transform two non-intersecting circles into two concentric circles, and as a special case ;

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*For a detailed study of circular triangles and quadrilaterals see:

A. Schönfliess: Ueber Kreisbozendreiecke und Kreisbogenvierecke, Math. Annalen, Vol. 44, p. 105.

Fr. Schilling: Beiträge zur geometrischen Theorie der Schwarz 'schen S-Function, Math. Annalen, Vol. 44, p. 161.

There are always two circular transformations, each consisting of a translation and an inversion, which transform two non-intersecting circles into two concentric circles.

To prove this it will be remembered that there always exists a translation so that the line connecting the centers of the two translated circles passes through the origin of the z-plane. This translation being made in the proper way and adding an inversion the new circles will be concentric. Without affecting the general result the centers of the original circles may be assumed on the x-axis, so that the equations of these circles in Cartesian coördinates are

$$\begin{aligned}(x - a_1)^2 + y^2 &= \rho_1^2 ,\\ (x - a_2)^2 + y^2 &= \rho_2^2 .\end{aligned} \tag{8}$$

Applying a translation

$$\begin{aligned}x &= x' + a ,\\ y &= y' ,\end{aligned} \tag{9}$$

the equations of the new circles become

$$\begin{aligned}x'^2 + y'^2 - 2\,(a_1 - a)\,x' + (a - a)^2 - \rho_1^2 &= 0 ,\\ x'^2 + y'^2 - 2\,(a_2 - a)\,x' + (a_2 - a)^2 - \rho_2^2 &= 0 .\end{aligned} \tag{10}$$

By an inversion $z' = \frac{1}{\bar{z}''}$, or

$$x' = \frac{x''}{x''^2 + y''^2} , \quad y' = - \frac{y''}{x''^2 + y''^2} \tag{11}$$

the circles (10) are transformed into two new circles, whose centers still lie on the x-axis. Hence the abscissæ of these circles are given by half the negative coefficients of x'' in the equations obtained from (10) by the inversion (11). These coefficients are, after dividing by -2,

$$\frac{a_1 - a}{(a_1 - a)^2 - \rho_1^2} \text{ and } \frac{a_2 - a}{(a_2 - a)^2 - \rho_2^2} .$$

The inversion makes the circles (10) concentric if

$$\frac{a_1 - a}{(a_1 - a)^2 - \rho_1^2} = \frac{a_2 - a}{(a_2 - a)^2 - \rho_2^2}\,.$$

From this a has to be found which, after some calculation, becomes

$$a = a_1 + \frac{(a_2 - a_1)^2 + \rho_1^2 - \rho_2^2 \pm \sqrt{[(a_2 - a_1)^2 + \rho_1^2 - \rho_2^2]^2 - 4\,(a_2 - a_1)^2 \rho_1^2}}{2\,(a_2 - a_1)}\,.$$

According to this expression there are two translations which when combined with an inversion transform any two circles into two concentric circles.

These translations are always real if

$$[(a_2 - a_1)^2 + \rho_1^2 - \rho_2^2]^2 - 4\,(a_2 - a_1)^2 \rho_1^2 > 0\,.$$

This condition may be reduced to the two following ones; provided $\rho_2 > \rho_1$,

If $|\,a_2 - a_1\,| > \rho_1$ there must be

$$|\,a_2 - a_1\,| > \rho_1 + \rho_2\,,$$

and if $|\,a_2 - a_1\,| < \rho_1$

$$|\,a_2 - a_1\,| < \rho_2 - \rho_1\,.$$

Transformations of this kind transform the region without the two original circles into the ring shaped domain of the transformed circles which are concentric. This important remark is illustrated in fig. 5.

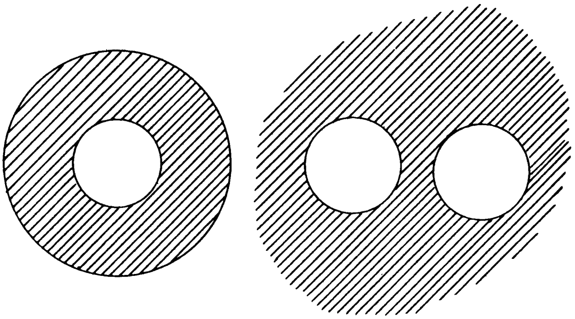

Fig. 5.

4. Two concentric circles do not change the properties relating to their mutual position by a transformation of similarity, since it leaves the ratio $\frac{r_1}{r_2}$ of the radii unchanged. Designating now the transformations by symbols in the usual manner, i. e., a translation by T, an inversion by I, and a transformation of similarity by S; furthermore two pairs of concentric circles whose radii form in each pair the same ratio by C_{12} and C_{12}'; there always exists a transformation ST so that

$$C_{12}' = C_{12}\, ST.$$

Making first a translation T_1, there is evidently

$$C_{12}' = C_{12} T_1 S,$$

hence

$$T_1 = STS^{-1}.$$

Considering any two non-intersecting circles, which we will represent by the symbol K_{12}, there always exists a transformation TI, so that

$$K_{12} TI = C_{12}$$

gives two fixed concentric circles. By means of two other transformations, a similarity S and a translation T_2, the pair of concentric circles C_{12} may be transformed into any other pair of concentric circles, C_{12}', provided the ratio $\frac{r_1'}{r_2'}$ of the radii of C_{12}' is the same as that of C_{12}. Hence

$$K_{12} TIST_2 = C_{12}'.$$

There are only two transformations $TIST_2$ which transform K_{12} into C_{12}'.

Adding to these transformations a rotation R through an angle φ, which is given by the formula

$$Z' = Ze^{\phi i}, \tag{12}$$

there always exists a translation T_3 so that

$$C_{12} RT_3 = C_{12}.\text{*}$$

The transformation RT_3 is a circular transformation which transforms the two concentric circles C_{12} into themselves; but the circles have been revolved

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* There is also a translation T by which

$$C_{12} TRT^{-1} = C_{12}.$$

about their common center through an angle φ. There are ∞^1 circular transformations RT_3 which transform C_{12} into itself, hence

$$K_{12}TIST_2RT_3 = C_{12}' .$$

These results may be stated in the theorems:

There are ∞^1 *circular transformations which transform two non-intersecting circles into the same two concentric circles and also* ∞^1 *circular transformations which transform two non-intersecting circles into themselves.*

If a circular transformation transforms two non-intersecting circles into two concentric circles whose radii r_1 *and* r_2 *form the ratio* $\frac{r_1}{r_2} = k$ *there are always* ∞^1 *circular transformations which transform the original two circles into any two concentric circles whose radii form the same ratio* k.

There is no circular transformation Γ *which transforms the two original circles into two concentric circles whose radii form a ratio different from* k.

For, let it be granted that Γ_1 is a circular transformation so that

$$K_{12}\Gamma_1 = C_{12}' ,$$

and the ratio of the radii of C_{12}' is equal to k, and assume that Γ_2 is a circular transformation by which

$$K_{12}\Gamma_2 = C_{12}''$$

represents two concentric circles whose radii form the ratio $k_1 \neq k$, then there must surely exist a circular transformation Γ_3 which gives the result

$$C_{12}'\Gamma_3 = C_{12}'' .$$

This, however, is impossible; hence there is no transformation Γ_3 effecting this result.

Taking any circular transformation Γ there is

$$K_{12}\Gamma = K_{12}',$$

where K_{12} represents two other non-intersecting circles. Let Γ_1 be a circular transformation which transforms K_{12}' into a pair of concentric circles, then

$$K_{12}\Gamma\Gamma_1 = K_{12}'\Gamma_1 = C_{12} .$$

From this it is seen that $\Gamma\Gamma_1$ is a circular transformation which transforms K_{12}' into the same pair of concentric circles as K_{12}' is transformed by Γ_1. The ratio of the radii of each pair of concentric circles derived from K_{12}' is, therefore, the same as that derived from K_{12}. Hence:

If two pairs of non-intersecting circles correspond to each other in a circular transformation there always exist two systems of ∞^1 *circular transformations which respectively transform the two pairs into the same pair of concentric circles whose radii bear the necessary ratio.*

If we now consider a pair of non-intersecting circles K_{12} and construct a series of tangent circles in such a manner that each of these is tangent to the preceding and following circle and to the given circles K_{12} (every circle of the series must touch the given circles K_{12} in the same manner) two cases are possible: either the series closes, or does not close, i. e., does or does not repeat itself. If it closes, the series consists generally of m tangent circles representing k entire revolutions in the series around the circles K_{12}. Evidently the numbers m and k are relative primes and $m > k$.

There are always circular transformations which transform the pair K_{12} into two concentric circles, whose radii bear a constant ratio λ for all such transformations, and which do not change the numbers m and k. By these transformations all circles of the series become equal. Designating the angle included by the tangents from the center of the concentric circles to one of the tangent circles by φ, there is $\varphi = \frac{2k\pi}{m}$. Assuming the radii of the concentric circles as r_1 and r_2 and $r_1 > r_2$ we have easily

$$\frac{\lambda - 1}{\lambda + 1} = \sin\frac{k\pi}{m}, \tag{13}$$

which shows directly that every circular transformation which leaves λ unchanged also leaves the numbers k and m unchanged.

Thus we have the theorems :

Two series of tangent circles which both close and have the same number m *of circles and the same number* k *of revolutions correspond to each other in a circular transformation.*

Every circular transformation leaves the numbers in m *and* k *unchanged.*

It has been stated that there are ∞^1 circular transformations which transform two given circles into themselves. Hence, by the previous theorem we are led to one of Steiner's peculiar theorems :*

If one series of tangent circles belonging to a pair of circles k_{12} closes and contains m circles and k revolutions, then all ∞^1 series of tangent circles belonging to k_{12} close and contain the same number m of circles and k revolutions.

As to the properties of relative primes m and k we have

$$\frac{2k\pi}{m} = \varphi \leqq \pi ,$$

hence

$$\frac{k}{m} \leqq \frac{1}{2} \ , \quad \text{or} \quad m \geqq 2k .$$

In the case that $m = 2k$, k becomes equal to 1, thus for values $k > 1$, the condition is

$$m \geqq 2k + 1 .$$

5. In this paragraph I shall develop the relation between what may be called two *conjugate series of tangent circles.* Such series are obtained in the following manner. Two non-intersecting circles may always be transformed into two concentric circles. This is also true for two non-intersecting circles of equal radii. Conversely two concentric circles can always be transformed into two non-intersecting circles of equal radii. Suppose Γ is such a circular transformation which transforms two concentric circles C_{12} into two non-intersecting circles K_{12} of equal radii. The numbers m and k of each series of tangent circles belonging to either C_{12} or K_{12} are equal. Now take two opposite tangent circles C^k_{12} and construct the series of tangent circles belonging to the two equal circles C^k_{12}, then I call any series of tangent circles obtained from one belonging to C_{12} by a circular transformation together with any series obtained from one belonging to C^k_{12} by a circular transformation *two conjugate series of tangent circles.*

To establish the relation of two series of this kind consider again the equation of a circle

$$zz_0 + Az + A_0 z_0 + B = 0 ,$$

whose radius is ρ and whose center has the Cartesian coordinates α and β. By an inversion from the origin $z = 0$ the equation of the inversed circle becomes

$$zz_0 + \frac{A_0}{B} z + \frac{A}{B} z_0 + \frac{1}{B} = 0 ,$$

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* Jacob Steiner's ges. Werke, Vol. I, p. 43.

and the coordinates of its center,

$$a = \frac{\alpha}{\alpha^2 + \beta^2 - \rho^2}, \quad b = -\frac{\beta}{\alpha^2 + \beta^2 - \rho^2}, \tag{14}$$

and the square of its radius

$$r^2 = \frac{\rho^2}{(\alpha^2 + \beta^2 - \rho^2)^2}. \tag{15}$$

The condition that two concentric circles C_{12} with the common center (α, β) and the radii ρ_1 and ρ_2 are transformed into two circles of equal radii is

$$\frac{\rho_1^2}{(\alpha^2 + \beta^2 - \rho_1^2)^2} = \frac{\rho_2^2}{(\alpha^2 + \beta^2 - \rho_2^2)^2},$$

which reduces to

$$\rho_1\rho_2 = \alpha^2 + \beta^2. \tag{16}$$

Choosing in fig. 6, $CP = \rho_1$ the magnitude of ρ_2 is found by drawing a tangent PT to the circle having the point (α, β), or C as a center and $CO = \sqrt{\alpha^2 + \beta^2}$

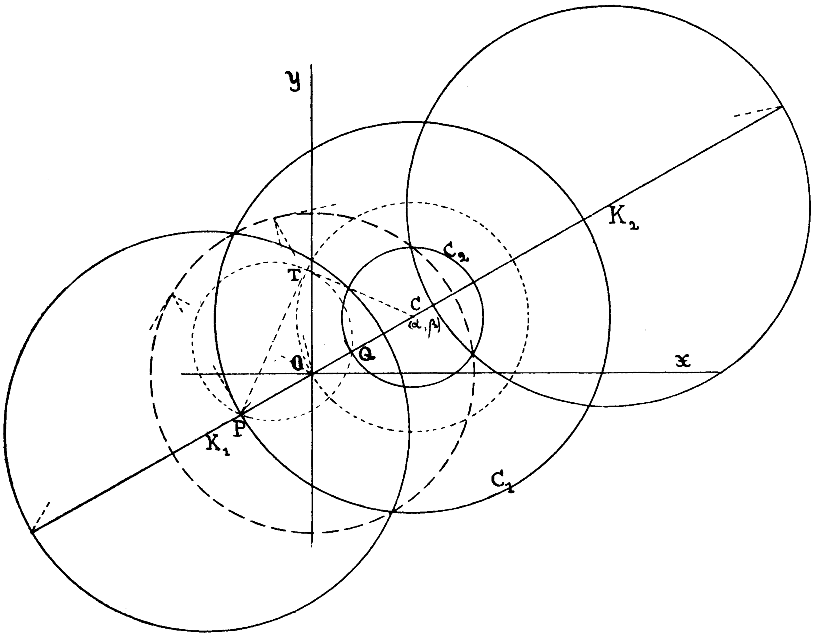

FIG. 6.

as a radius. The circle over PT as a diameter intersects CP in a point Q and from the construction there is evidently $CQ = \rho_2$.

If ρ_1 and ρ_2 are given in the first place, the distance of the point c from the origin $z = 0$ is determined by $\alpha^2 + \beta^2 = \rho_1 \rho_2$. The square of the radius of the two equal circles K_{12} corresponding to the concentric circles C_{12} in the inversion is

$$r^2 = \frac{1}{(\rho_1 - \rho_2)^2}, \qquad (\rho_1 > \rho_2) \tag{17}$$

and the Cartesian coordinates of their centers

$$a_1 = -\frac{\alpha}{\rho_1(\rho_1 - \rho_2)}, \quad b_1 = \frac{\beta}{\rho_1(\rho_1 - \rho_2)},$$

$$a_2 = \frac{\alpha}{\rho_2(\rho_1 - \rho_2)}, \qquad b_2 = -\frac{\beta}{\rho_2(\rho_1 - \rho_2)}.$$

From this we find for the square of the distance between the centers of circles C_{12}^k

$$(a_2 - a_1)^2 + (b_2 - b_1)^2 = \frac{(\rho_1 + \rho_2)^2}{(\rho_1 - \rho_2)^2}$$

or for the distance D itself

$$D = \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2}. \tag{18}$$

With the middle-point M of the geometrical distance $D = \overline{K_1 K_2}$ as a center describe two concentric circles C_{12}^k both tangent to the circles K_{12}. It is clear that the pairs of circles C_{12}^k and K_{12} are the fundamental circles of two conjugate series of tangent circles, hence also the pairs C_{12} and C_{12}^k. The radii of the concentric circles C_{12}^k are

$$R_1 = \frac{D}{2} + r,$$

and

$$R_2 = \frac{D}{2} - r,$$

or

$$R_1 = \frac{\rho_1 + \rho_2 + 2}{2(\rho_1 - \rho_2)}, \quad R_2 = \frac{\rho_1 + \rho_2 - 2}{2(\rho_1 - \rho_2)}.$$

Designating the angle included by the tangents from M to one of the circles K_{12} by ψ, there is according to formula (13)

$$\sin\frac{\psi}{2} = \frac{R_1 - R_2}{R_1 + R_2} = \frac{2}{\rho_1 + \rho_2}\,. \tag{19}$$

This equation defines the angle ψ relative to the conjugate series $C_{12}{}^k$ of C_{12}. For C_{12} we have found

$$\sin\frac{\varphi}{2} = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\,. \tag{20}$$

I will now form the expression

$$\left[\sin\frac{\varphi}{2}\right]^2 + \left[\sin\frac{\psi}{2}\right]^2 = \frac{(\rho_1 - \rho_2)^2 + 4}{(\rho_1 + \rho_2)^2}\,, \tag{21}$$

which holds for any positive real ratio $\frac{\rho_1}{\rho_2}$. The condition that the right side of (21) is equal to unity is

$$4 + (\rho_1 - \rho_2)^2 = (\rho_1 + \rho_2)^2\,,$$

or

$$\rho_1\rho_2 = 1\,.$$

Taking for ρ_1 any real positive value, then $\rho_2 = \frac{1}{\rho_1}$, and the ratio $\frac{\rho_1}{\rho_2} = \rho_1{}^2$ can be any positive real number. The assumption $\rho_1\rho_2 = 1$ does, therefore, not limit the system of all series of tangent circles, and includes by circular transformations all series of tangent circles. We have, therefore, for two conjugate series of tangent circles the relation

$$\frac{\psi}{2} = \frac{\pi}{2} - \frac{\varphi}{2}\,,$$

or

$$\psi = \pi - \varphi\,. \tag{22}$$

Assuming φ as a rational divisor of 2π, i. e., $\varphi = \frac{2k\pi}{m}$, where k and m have the same meaning as before, there is

$$\psi = \pi - \frac{2k\pi}{m} = \frac{2\,(m - 2k)\,\pi}{2m}$$

Denoting by $\frac{(m\,2k)'}{(2m)'}$ the reduced fraction of $\frac{m - 2k}{2m}$, $(2m)'$ is the number of circles and $(n - 2k)'$ the number of revolutions in the series of tangent circles $C_{12}{}^k$ conjugate to the series C_{12}, hence the theorem :

If a series of tangent circles closes, any of its conjugate series closes also.

Between the numbers of circles and revolutions in two conjugate series of tangent circles exists the remarkable relation :

$$\frac{m-2k}{2m}+\frac{k}{m}=\frac{1}{2},$$

or in words :

The sum of the ratios formed in each of any two closing conjugate series of tangent circles by the number of revolutions into the number of circles is equal to $\frac{1}{2}$.*

As an example take $k=1$ and $m=3$, then $\frac{m-2k}{2m}+\frac{1}{3}=\frac{1}{2}$, hence $\frac{m-2k}{2m}=\frac{1}{6}$.

A conjugate series has one revolution and six circles. In fig. 7 the circles of the original series are drawn in solid lines, while those of the conjugate series are represented by dotted lines.

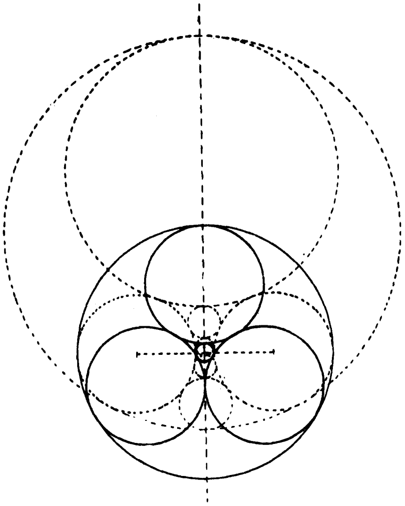

FIG. 7.

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* Similar results are obtained for series of circles of which two consecutive angles include a constant angle.

To find two conjugate series, each having the same numbers of revolutions and circles, put

$$\frac{n-2k}{2n}=\frac{k}{n},$$

from which $n-2k=2k$, or $n=4k$, hence the ratio $\frac{k}{n}=\frac{1}{4}$.

Every conjugate series of a series containing four circles contains also four circles.

This case is illustrated in fig. 8.

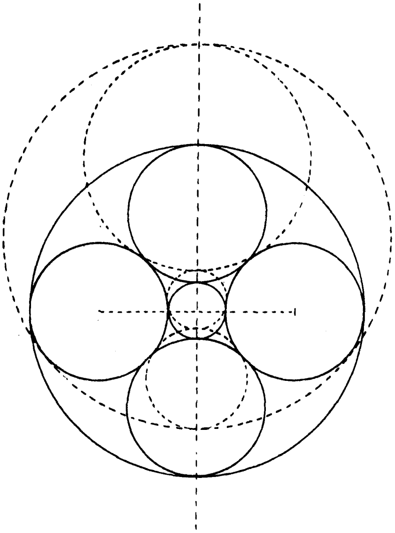

FIG. 8.

6. The properties of series of tangent-circles in the plane can easily be transferred to series of tangent spheres in space. If in a tangent series of circles in the plane every circle is considered as the intersection of this plane with a sphere whose center is in this plane, a series of tangent spheres arises which are all tangent to a *Drysinian cyclide* and we have immediately the theorem:

If among the enveloping spheres of a Drysinian cyclide a series of tangent spheres may be formed which closes and contains m spheres and k revolutions, then an infinite number of such series may be formed by the same generating spheres of the cyclide all of which close and each contains the same number m of spheres and k revolutions.

Taking any three spheres out of these tangent series, the same *Drysinian cyclide* is also obtained as the envelope of all spheres tangent to these three spheres. In this manner the two systems of generating, or enveloping, spheres of a Drysinian cyclide arise. Every enveloping sphere touches the cyclide along a circle. The two systems of these circles corresponding to the two systems of enveloping spheres form the orthogonal system of lines of curvature of the Drysinian cyclide. Remembering these well known facts and remarking that every sphere of one system of enveloping spheres touches every sphere of the other system, the theorem which I have established concerning two conjugate series of tangent circles can immediately be generalized.

Any series of tangent spheres formed out of the spheres of one system of enveloping spheres is conjugate to any series of tangent spheres out of the other system.

If the ratio of the number of revolutions into the number of spheres in a series of the first system that closes is $\frac{k_1}{m_1}$, *then every tangent series of the other system closes, and the ratio* $\frac{k_2}{m_2}$ *belonging to these series is such that*

$$\frac{k_1}{m_1} + \frac{k_2}{m_2} = \frac{1}{2}.\text{*}$$

This is one of the most remarkable properties of a Drysinian cyclide. It can also be derived from the consideration of a torus, for which the theorem formulated for the plane holds, when the two concentric circles are replaced by a torus and the system of equal circles by spheres each tangent to the torus along a meridian circle. By a conform transformation in space, *which is always circular*, the torus with its two systems of enveloping spheres is transformed into a Drysinian cyclide and the number-relations for conjugate series of tangent circles in the plane, which are the same for the conjugate tangent series of the torus, are immediately transplanted to tangent-series of spheres belonging to the Drysinian cyclide.

I will pass over a great number of applications and details derived from the general theorems that I have stated and give finally some references, concerning the cyclides and similar subjects.

1. Darboux: *Sur une Classe Remarquable de Courbes et de Surfaces Algébriques et sur la théorie des Imaginaires. Seconde tirage. Paris, A. Hermann*, 1896.

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* This theorem, Steiner, in loc. cit., p. 136, gives in a special form without proof and not in connection with the surfaces now called Drysinian cyclides.

An account of this memoir is given in No. 8, Vol. II, 2nd series of the bulletin of the Am. Math. Soc. by R. A. Roberts.

2. ....Leçons sur la Théorie Générale des Surfaces. Vol. I, chpts. V and VI in Livre II.

3. Sophus Lie, *Über Complexe, insbesondere Linien—und Kugel—Complexe, mit Anwendung auf die Theorie partieller Differentialgleichungen.* Math. Ann., Vol. V, p. 145–256; 1871.

4. ....*Geometrie der Berührungs-transformationen, Vol. I, Leipzig, Teubner*, 1896.

The study of series of tangent circles and spheres whose circles and spheres remain finite but do not close is reduced to the theory of an approximate representation of numbers by rational functions. Thus, any value for the angle φ being given the numbers k and m must be determined in such a manner, that in the expression

$$\frac{2k\pi}{m} = \varphi + \Sigma$$

the quantity Σ becomes as small as possible. If the integer m is given in advance as a limit of the denominator in the approximation, the problem may be solved by the so-called *series of Farey* for which particular reference must be made to an article of Prof. A. Hurwitz in Zurich in the Mathematische Annalen. *Ueber die angenäherte, Darstellung der Zahlen durch rationale Brüche.*

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